

# Digital topology and finite approximation

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## 1: Adjacencies

Digital topology necessarily tries to get good finite representations of infinite spaces. Then it analyzes them, often using results and intuition about Euclidean space and polyhedra. Convenient algorithms for doing this are often written in terms of adjacency, going from point to adjacent point.

The traditional 2-dimensional adjacencies are: *4-adjacency*, in which a point is adjacent to the four which are immediately above, below, to the left, and to the right, of it, and *8-adjacency*, in which it is adjacent to the above four, and the four which are diagonally one away from it. So for a point,  $p$ , we have the diagram:

$$\begin{array}{ccccc} & 8 & 4 & 8 & \\ & 4 & p & 4 & \\ & 8 & 4 & 8 & \end{array}$$

An important problem in such work is the replacement of regions by their boundaries; this can result in considerable data compression. In the Euclidean plane, the Jordan curve theorem is the key tool. Recall that a *Jordan curve* is a homeomorphic (= continuous one-one, inverse continuous) image of the circle; equivalently, it is a continuous image of  $[0, 1]$  under a map which is one-one on  $[0, 1)$  and  $f(0) = f(1)$ . Then:

**Jordan curve theorem.** *If a Jordan curve,  $J$ , is removed from the plane, the remainder,  $\mathbb{R}^2 \setminus J$  consists of two connected components.*

Here, a set is *connected* if it is not the union of two nonempty subsets, neither of whose closures meets the other. A *connected component* of a set is a maximal connected subset.

A problem is how to translate this into the finite situation, and with adjacencies rather than topology. In this case, given an adjacency  $\alpha$  on a digital plane, it seems reasonable to define an  $\alpha$ -*Jordan curve* as a set  $J$  such that each element of  $J$  is adjacent to exactly two others in  $J$ .

The other topological concept, connectedness, also has an adjacency replacement: a set  $S$  is  $\alpha$ -*path connected*, if for each  $x, y \in S$  there is an  $\alpha$ -*path* from  $x$  to  $y$ : a sequence  $x_0, \dots, x_n$  so that  $x = x_0$ ,  $y = x_n$ , and if  $j < n$ ,  $x_j$  is adjacent to  $x_{j+1}$ .

Also, an  $\alpha$ -*arc* from  $x$  to  $y$  is an  $\alpha$ -*path* from  $x$  to  $y$  in which if  $x_j$  is adjacent to  $x_k$ , then  $k = j \pm 1$ , and an  $\alpha$ -*Jordan curve* is an  $\alpha$ -path in which  $x_0 = x_n$  and if  $x_j$  is adjacent to  $x_k$ , then  $k = j \pm 1 \pmod n$ .

But the following pictures of portions of  $\mathbb{Z}^2$  show a Jordan 8-curve which doesn't 8-disconnect it and a 4-curve breaking it into 3 4-components:

$$\begin{array}{cccccccc} \cdot & \cdot & x & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & x & \cdot & x & \cdot & \cdot & \cdot & x & x & x & \cdot & \cdot \\ x & \cdot & \cdot & \cdot & x & \cdot & \cdot & x & \cdot & x & x & \cdot \\ \cdot & x & \cdot & x & \cdot & \cdot & \cdot & x & x & \cdot & x & \cdot \\ \cdot & \cdot & x & \cdot & \cdot & \cdot & \cdot & \cdot & x & x & x & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

**(Rosenfeld) Jordan curve theorem.** *If a Jordan  $k$ -curve with  $\geq 5$  points,  $J$ , is removed from the digital plane, the remainder,  $\mathbb{Z}^2 - J$  consists of two  $k'$ -connected components, for  $\{k, k'\} = \{4, 8\}$ .*

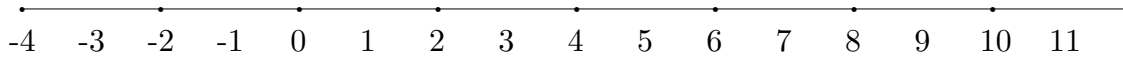
The slight artificiality seen in these pictures and results is easily forgiven to write algorithms, but today we look at how to avoid it.

Below we look at topologies on the finite set of pixels on a computer screen.

## 2: Digital $n$ -Space and finite topological spaces

Looking again at the pictures, we see a way to define a one-dimensional digital space: A *connected ordered topological space (COTS)* is a connected space such that among any 3 points is one whose deletion leaves the other two in separate components of the remainder.

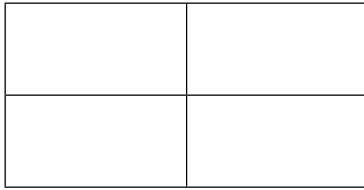
The *digital line* is the integers,  $\mathbb{Z}$ , with the topology in which a set is open if and only if it is a union of sets of the form  $\{2n - 1, 2n, 2n + 1\}$  and  $\{2n + 1\}$ . A *digital interval* is an interval  $[p, q] = \{p, p + 1, \dots, q\}$ ,  $p < q$ , with the subspace topology.



Digital topology has taught us how to visualize finite topological spaces, using two conventions which enable us to draw “Euclidean” pictures and interpret them as finite topological spaces:

- apparently featureless sets represent points,
- sets which ‘look’ open are open.

The above picture is interpreted using these conventions. Here is a two-dimensional space (the product of two finite COTS) with 4 open points, 9 closed points, and 12 which are neither:



Similarly, a 2-simplex (triangle) can be viewed as a set with 7 points: the ‘2-dimensional’ interior, the 3 ‘1-dimensional’ (open segment) edges and the 3 (closed) ‘0-dimensional’ vertices. A 3-simplex can be represented by an apparent tetrahedron in  $\mathbb{R}^3$ , which is seen as having 15 points: 1 open (the ‘interior’), 4 faces, 6 edges and 4 vertices. It should be noted that the conventions do not lead to a unique picture of a finite space.

*Digital  $n$ -space* is the product of  $n$  copies of the digital line,  $\mathbb{Z}^n$ , and a *digital  $n$ -box* is the product of  $n$  digital intervals. In particular the *digital plane* is digital 2-space, and a *digital box* is a digital 2-box.

The finite spaces we use are connected, so not  $T_1$ ; it is possible for  $x$  to be in  $\text{cl}\{y\}$ ; for example,  $2n, 2n + 2 \in \text{cl}\{2n + 1\}$ . If  $x \in \text{cl}\{y\}$  or  $y \in \text{cl}\{x\}$ ,  $x \neq y$ , we say that  $x, y$  are adjacent. Thus, we have the advantages of topology and those of adjacency:

A *digital Jordan curve* is the image of a digital interval  $[p, q]$  such that  $q > p + 2$ , under a function  $J$  which would be a homeomorphism, except that  $J(p), J(q)$  are adjacent. Equivalently, it is a set  $J$  such that each element of  $J$  is adjacent to exactly two others in  $J$ . Also, a *digital path* from  $x$  to  $y$  is a continuous image of an interval  $[p, q]$ , or equivalently, sequence  $x_0, \dots, x_n$  so that  $x = x_0$ ,  $y = x_n$ , and for  $j < n$ ,  $x_j$  is adjacent to  $x_{j+1}$ , and a

set  $S$  is *digitally path connected*, if for each pair  $x, y \in S$  there is a digital path from  $x$  to  $y$ . Then we have:

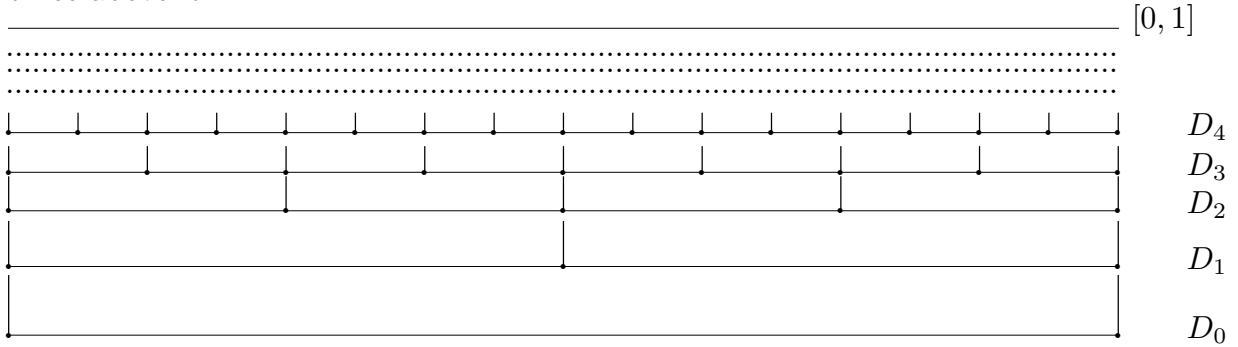
**Digital Jordan curve theorem.** *If a digital Jordan curve  $J$  is removed from the digital plane (or a digital box whose border it does not meet), the remainder consists of two connected components.*

In this situation, we retain the advantage that algorithms can be written in terms of adjacency, in a topological space that acts very much like the plane.

### 3: Finite approximation

It has long been known that each compact  $T_2$  space is the  $T_2$  reflection (“largest”  $T_2$  continuous image) of an inverse limit of finite  $T_0$ -spaces ([F1], [KW1]). The next diagram suggests such a way to approximate the unit interval by finite spaces. Its top horizontal line represents the unit interval, but those at the bottom are meant to be finite COTS:

$D_n = \{\frac{i}{2^n} \mid 0 \leq i \leq 2^n\} \cup \{(\frac{i}{2^n}, \frac{i+1}{2^n}) \mid 0 \leq i < 2^n\}$ , with  $2^{n+1} + 1$  points and the quotient topology induced from  $[0, 1]$ . The vertical lines indicate maps going down, for which a closed point is the image of the one directly above it, while an open point is that of the three above it:



The above mappings are special:

**Definition.** A continuous map is *normalizing* if inverse images of disjoint closed sets are contained in disjoint open sets.

We knew our constructions used such maps and gave inverse limits whose Hausdorff reflections were their subspaces of closed points.

The first definition is unsurprising to topologists: Hausdorff normal spaces are those most familiar; they include all metric and all compact  $T_2$  spaces. Finite  $T_0$  spaces are typically non-normal, but this is partly made up using normalizing maps:

**Theorem.** *An inverse system of finite  $T_0$ -spaces and continuous maps is normalizing if and only if its inverse limit is normal.*

**Definition.** A  $T_0$  space  $(X, \tau)$  is *skew compact* if there is a second topology  $\tau^*$  on  $X$  such that

$$x \in \text{cl}\{y\} \Leftrightarrow y \in \text{cl}^*\{y\},$$

whenever  $x \notin \text{cl}\{y\}$  then there are disjoint  $T \in \tau, U \in \tau^*$  such that  $x \in T$  and  $y \in U$ , and

$$\tau \vee \tau^* \text{ is compact.}$$

Then the Hausdorff reflection is easy to see:

**Theorem.** ([KW2]) *The following are equivalent for a skew compact space  $X$ :*

- (a)  $X$  is normal,
  - (b) each point of  $X$  has a unique closed point in its closure (that is,  $\mu = \{(x, y) \mid y \in \text{cl}\{x\}, \{y\} \text{ closed}\}$  is a function),
  - (c)  $\mu$  is a retract from  $X$  to  $\mu[X]$  (its subspace of closed points).
- If any of these hold, then  $(\mu(X), \mu)$ , is the Hausdorff reflection of  $X$ .

**Theorem.** *A topological space is compact Hausdorff if and only if it is the space of minimal points of an inverse limit of finite spaces and normalizing maps. It is compact Hausdorff and connected if and only if it is the space of minimal points of an inverse limit of such such a system in which the finite spaces are also connected.*

### References

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